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Times and Sizes of Jumps in the Mexican Interest Rate

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Abstract

This paper examines the role of jumps in a continuous-time short-term interest rate model for Mexico. A filtering algorithm provides estimates of jumps times and sizes in the time series of Mexican CETES for the 1998-2006 period. The empirical results indicate that the inclusion of jumps in the diffusion model represents a better alternative than not to include them.

Key words: Jumps, Monte Carlo, Diffusion model, Gibbs sampler.
JEL Classification: C22, G0.

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Introduction

Many economic and financial models make use of an interest rate in some form. A model with a good description and forecasting of interest rates can be used for example in providing parallel structure correct valuation of financial derivatives, giving a better explanation of the future performance of the economy and describing more accurately the weighted average cost of capital (WACC). Specification of the interest rate dynamics is an important issue in finance, economics and accounting.

Many of the continuous time models used to specify the dynamics of short-term interest rates have omitted the possibility of jumps, but typical models which only include Brownian motion are not able to explain some facts like fat tails and skewness. To get an idea we can see in figure 1 that the daily changes in interest rates show some spikes, which could be interpreted as jumps. The changes in the short-term interest rate go from -13.93% to 15.47% from 11-04-1998 to 04-19-2006.

The goal of this paper is to get the times and sizes of jumps in the dynamics of interest rates (CETES). A tractable way to do the analysis is including Poisson jumps in the diffusion model like in Johannes(2004) model, which has been applied for the Mexican data in Núñez and Lorenzo(2007) where the levels of conditional and unconditional kurtosis, generated by a pure diffusion model, were compared with the sample conditional and unconditional kurtosis from a short-term rate. In those papers, a nonparametric specification for the drift, volatility and jump intensity was used with log normal jump size. For the Mexican case it was demonstrated that including jumps in the diffusion process provides a better representation of the data than the pure diffusion model. References in the literature can be found about the issue of using or not diffusion continuous models of interest rates or including jumps in a parametric context (see Ait-Sahalia, 1996).

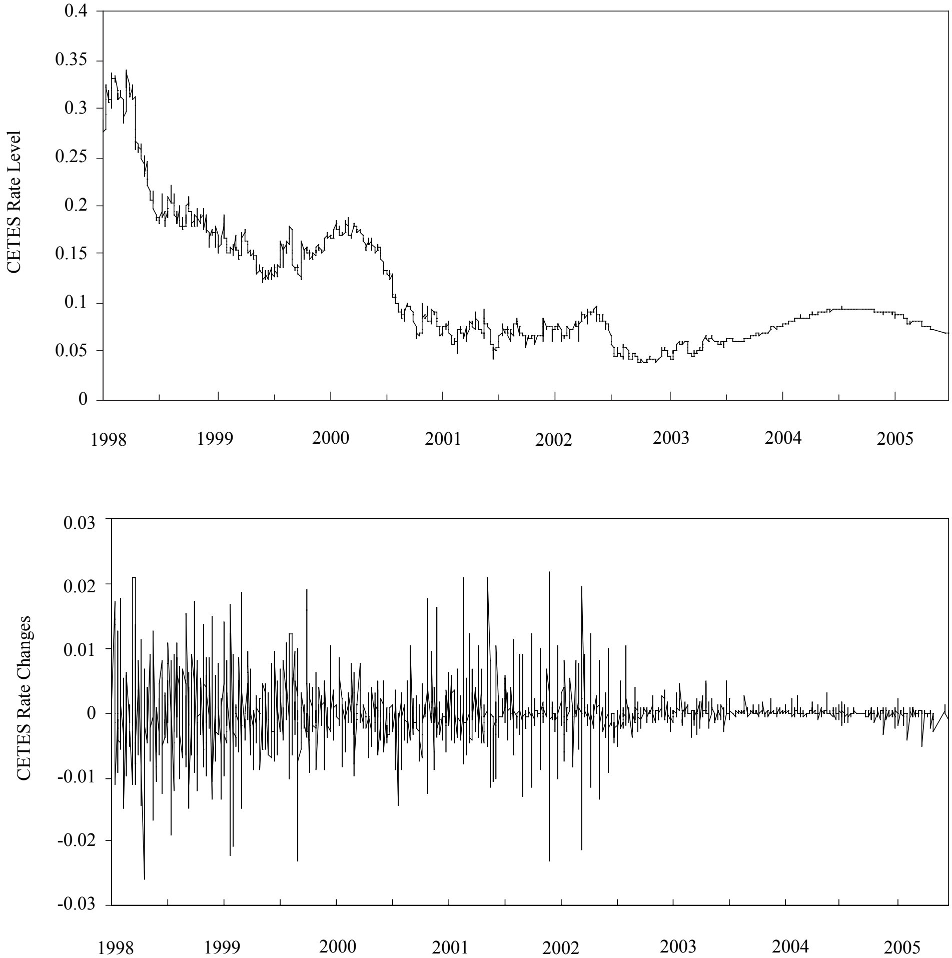
We develop the jump-diffusion model of the short-term interest rate in the following section. We display the empirical estimation for the Mexican case in the third section and establish some conclusions in the fourth.

1. Diffusion models and jumps

Pure diffusion models of the interest rate which include Brownian motion are written in the standard form:

$$dr_1 = \mu(r_t)dt + \sigma(r_t)dW_t \tag{1}$$

Figure 1
The time series of the level and changes in the CETES rate, november 1998 to april 2006



Source: Reuters.

where W_t is a standard Brownian motion defined on Ω, F, $\{F_t\}_{t\geq 0}$,P) with $\{F_t\}$ a filtered space, and (μ, σ) are the drift and volatility respectively. Under technical regularity conditions (Jacod and Shiryaev, 1987):

$$\lim_{\Delta\to 0}\frac{1}{\Delta}E[(r_{t+\Delta}-r_t)|r_t)=\mu(r) \quad (2)$$

$$\lim_{\Delta\to 0}\frac{1}{\Delta}E[(r_{t+\Delta}-r_t)^2|r_t]=\sigma^2(t) \quad (3)$$

and in the case of a pure diffusion model it can be demonstrated that

$$\lim_{\Delta\to 0}\frac{1}{\Delta}E[(r_{t+\Delta}-r_t)^j|r_t]=0,\ j>2 \quad (4)$$

Thus, with the first two conditional moments is not possible to recognize the difference between a pure diffusion model and a jump diffusion model.

The jump diffusion model we propose is

$$d\log(r_t)=\mu(r_{t-})dt+\sigma(r_{t-})dW_t+\eta dJ_t \quad (5)$$

where $\eta\sim N(0,\sigma^2{}_\eta)$, the jump size, is a normal variable. J_t is a time-homogeneous pure-jump process and we are modeling the logarithm of the interest rate. In this sense, we avoid negative values of the interest rates like in Das (1998), Das (2001) and Zhou (1999).

In this case (the basic principles are in Gihman and Skorohod 1972, and Johannes 2004),

$$\lim_{\Delta\to 0}\frac{1}{\Delta}E[\log(r_{t+\Delta}/r_t)\mid r_t=r]=\mu(r) \quad (6)$$

$$\lim_{\Delta\to 0}\frac{1}{\Delta}E[\log(r_{t+\Delta}/r_t)^2\mid r_t=r]=\sigma^2(r)+\lambda(r)E[\eta^2] \quad (7)$$

$$\lim_{\Delta\to 0}\frac{1}{\Delta}E[\log(r_{t+\Delta}/r_t)^j\mid r_t=r]=\lambda(r)E[\eta^j],\ j>2 \quad (8)$$

Fourth and sixth moments are used for the identification of the intensity, λ(r), and variance $\sigma^2\eta$ of the jump process mean:

$$\lim_{\Delta\to 0}\frac{1}{\Delta}E_{t,r}[\log(r_{t+\Delta}/r_t)^4]=3\lambda(r)(\sigma^2{}_\eta)^2 \quad (9)$$

$$\lim_{\Delta \to 0} \frac{1}{\Delta} E_{t,r}[\log(r_{t+\Delta} / r_t)^6] = 15\lambda(r)(\sigma^2_\eta)^3 \tag{10}$$

We calculate the fourth and sixth moments, and with the quotient of both moments we get the volatility of the jump process (applying Monte Carlo integration). After this, we can obtain the intensity of the Poisson process.

The Euler discretization used for the simulation of paths (1000) is:

$$\log(r_{t+\Delta} / r_t) = \hat{\mu}(r_t)\Delta + \hat{\sigma}(r_t)(W_{t+\Delta} - W_t) + \eta_{t+\Delta} J_{t+\Delta} \tag{11}$$

and at the same time, Monte Carlo simulations give sample standard errors.

The steps to be followed in order to evaluate if certain set of observations were generated by a given diffusion model (Johannes, 2004) are:

1. Estimate the drift and diffusion coefficients of a given diffusion model.
2. Estimate nonparametrically $kurt_\Delta(r)$ from data.
3. Simulate a large number of paths from the diffusion model and calculate $kurt_\Delta(r)$ nonparametrically from each path.
4. Compare the kurtosis from observed data against the one from the diffusion model.

Where the conditional kurtosis of the short-term interest rate over the time interval Δ is

$$kurt_\Delta(r) = \frac{\frac{1}{\Delta} E[(r_{t+\Delta} - r_t)^4 \mid r_t = r]}{\frac{1}{\Delta} E\ [(r_{t+\Delta} - r_t)^2 \mid r_t = r]} \tag{12}$$

2. Estimation for the Mexican case

The data used in the analysis are daily changes of the short-term interest rate from November 4, 1998 through March 4, 2006. The estimates of the drift and diffusion on a given time interval Δ, with the kernel, is obtained by means of the equation

$$\frac{1}{\Delta} \hat{E}[(r_{t+\Delta} - r_t)^j \mid r_t = r] = \frac{\sum_{i=1}^{T} K\left(\frac{r_{(i+1)\Delta} - r}{h}\right) \frac{(r_{(i+1)\Delta} - r_{i\Delta})^j}{\Delta}}{\sum_{i=1}^{T} K\left(\frac{r_{(i+1)\Delta} - r}{h}\right)} = \begin{cases} \hat{\mu}(r) : j = 1 \\ \hat{\sigma}^2(r) : j = 2 \end{cases} \tag{13}$$

Here K(.) is a kernel (Gaussian in this case), and h is the bandwidth. We take different bandwidths for the estimation of the drift and diffusion coefficient. Note that the moments are functions of the interest rate r, because they are conditional on such r.

As a first idea we checked the summary statistics reported in table 1. We can see a big unconditional kurtosis of the short-term rate increments. This is an important comparative point between a simple diffusion and the jump diffusion model.

Table 1
Summary Statistics
Summary statistics for daily CETES rates from november 1998 to april 2006

	Mean	*Standard Deviation*	*Skewness*	*Kurtosis*	*First Autocorrelation*
r_1	0.11118	0.06296	1.48273	5.07239	0.99551
$r_t + \Delta\text{-}r_t$	-0.00012	0.00418	0.25794	10.46498	-0.06754

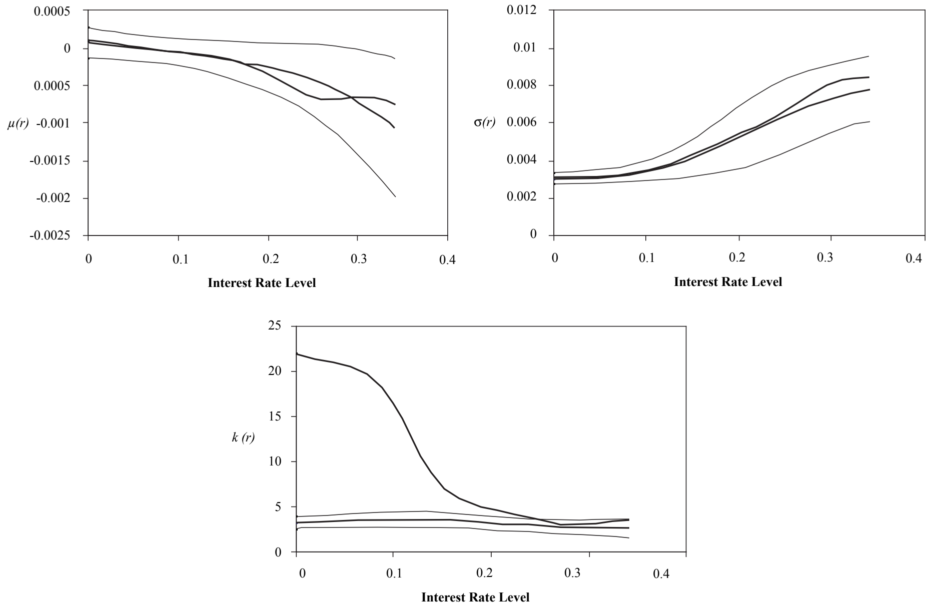
Source: Authors' calculations with data from INEGI.

2.1 Results of the pure diffusion model

We show the estimates of the drift and diffusion and the Monte Carlo standard errors in Figure 2, from Núñez and Lorenzo (2007). For the case of the drift we used h=0.05, the diffusion was calculated with h=0.04 and the fourth moment was calculated with h=0.05. One may notice the very good approximations for the drift and diffusion coefficients in the top left and right panels. However, in the case of the conditional kurtosis in the panel below there is a bad approximation of the model with respect to the data. The kurtosis from the data is approximately between four and two times greater than the kurtosis generated by the model at low and middle levels of interest rates. These same conclusions are obtained with different bandwidth choices

Then, the pure diffusion model is misspecified due to the fact that the model is unable to describe the kurtosis of the data. In conclusion, the pure diffusion model does not generate enough large movements to properly describe the data.

Figure 2
Nonparametric estimation results for the single-factor diffusion model. The solid line is the function estimated from short-term rate data, the thin solid line is the Monte Carlo median, and the dash lines are the confidence bands



Source: Authors' calculations.

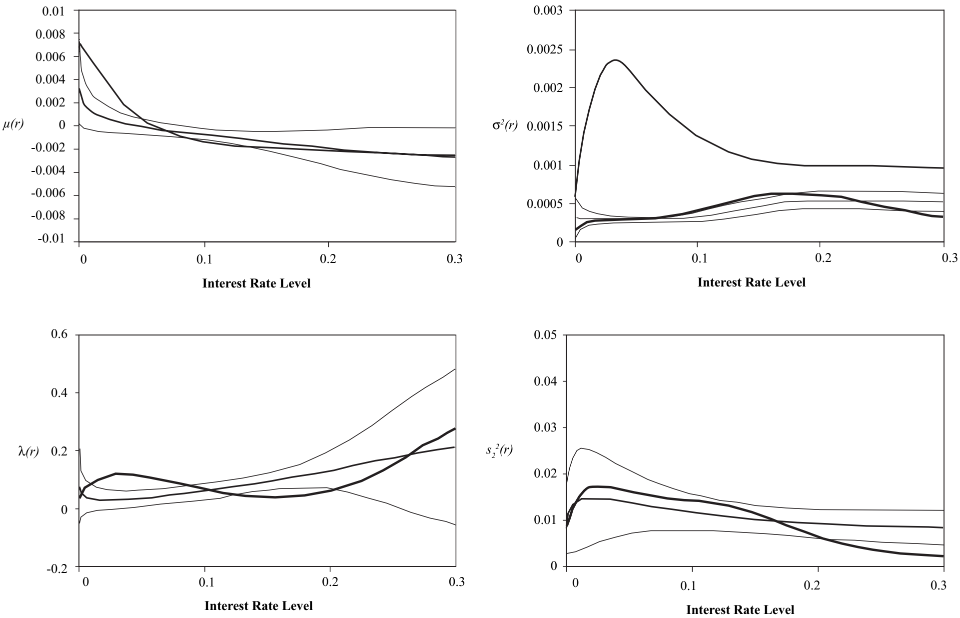
This led us to consider models with jumps. Johannes (2004) extended the nonparametric estimation methods from others, like Florens-Zmirou (1993), Stanton (1997) and Jiang&Knight (1998).

The Euler discretization used for the simulations of paths (1000) is:

$$\log(r_{t+\Delta}/r_t) = \hat{\mu}(r_t)\Delta + \hat{\sigma}(r_t)(W_{t+\Delta} - W_t) + \eta_{t+\Delta} J_{t+\Delta} \tag{14}$$

and at the same time, Monte Carlo simulations give sample standard errors. J_t indicates the presence of a jump, and $P[J_t = 1] = \lambda_t$, the intensity of the process.

Figure 3
Nonparametric estimation results for the jump-diffusion model. The solid line is the function estimated from short-term rate data, the thin solid line is the Monte Carlo median, and the dash lines are the confidence bands



Source: Authors' calculations.

2.2 Results of the jump diffusion

Note in figure 3, in the top left and right respectively, that the calculated levels of the drift and variance are very good approximations with different interest rates. There, the bandwidth used was $h = s$ for the drift, and the second, fourth and sixth moments were calculated with $h = 0.75s$, where s is the standard deviation of the data, (in this case $s = 0.5016$ is the standard deviation of $\ln(r)$).The fourth line in the top right is the sum of the variance of the process and the variance of the jumps. The rate of jumps in the left bottom is a good approximation even when we have certain levels of the rate in which the rate of the model does not enter in the confidence band. The bandwidth used was h=0.75s because it depends on the sixth and fourth moments.

For the variance of the process of jumps in the bottom right, the variance of the model has a very good level for the low and middle levels of the interest rates. At high levels we can say that they are good approximations.

2.3 Estimation of Jump Times and Sizes

Estimation of the times and sizes of jumps are provided with the help of the Gibbs algorithm. Gibbs sampling is the simplest Markov Chain Montecarlo Method (MCMC) for exploring the posterior distributions generated by continuous-time pricing models. MCMC uses the samples of estimated parameters generated by numerical integration. If all the conditional distributions can be directly sampled it is referred as a Gibbs sampler.

Consider the discretization:

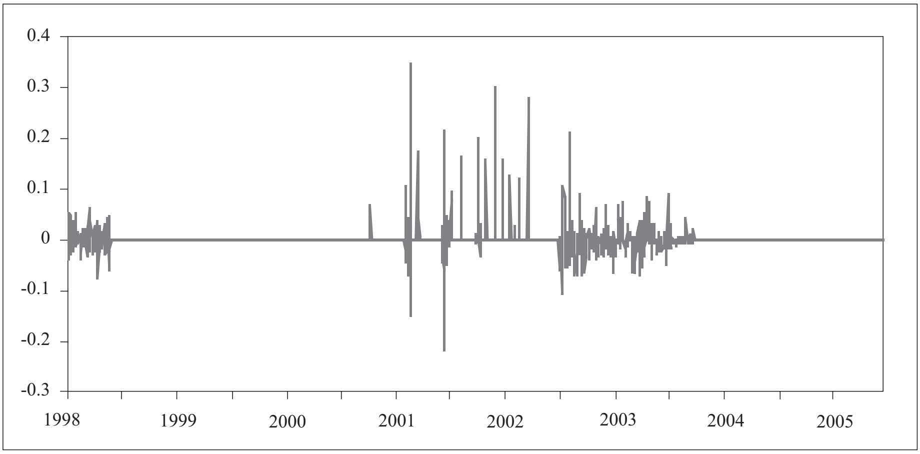
$$\log(r_{t+\Delta}/r_t) = \hat{\mu}(r_t)\Delta + \hat{\sigma}(r_t)\sqrt{\Delta}\varepsilon + \eta_{t+\Delta}J_{t+\Delta} \qquad (15)$$

where $P\left[J_{t+\Delta}=1\middle|r_t\right] = \hat{\lambda}(r_t)\Delta$ and $\varepsilon\succ N(0,1)$. With $\hat{\theta}_t = \left\{\hat{\mu}(r_t),\hat{\sigma}(r_t),\hat{\lambda}(r_t),\hat{\sigma}_\eta^2\right\}$.
From Gibbs' sampler we get (see Appendix 2, Johannes(2004)):

$$p(J_{t+\Delta}=1\,|\,\Delta r_t, r_t\hat{\theta}_t, \eta_{t+\Delta}) \quad \text{and} \quad p(\eta_{t+\Delta}J_{t+\Delta}\,|\,\Delta r_t, r_t, \hat{\theta}_t, J_{t+\Delta}=1) \qquad (16)$$

Applying the Gibbs sampler for 5000 iterations and eliminating the initial 2000 iterations, (burn-in period), we have obtained the results of figure 4. In this figure we show the times of the jumps (with the dates) and their respective size, which can be compared with the graph of the time series of the data.

Figure 4
Jump Times and their respective sizes



Source: Authors' calculations

Finally, we give a table with the size of the jumps, the date and the probability of jump.

Table 2
Size of jumps and their probabilities

Date	*Size*	*Probability*
21/12/2001	0.34805	0.11443
11/04/2002	-0.21564	0.11101
12/04/2002	0.21384	0.11854
02/08/2002	0.20099	0.11075
27/09/2002	0.30038	0.10339
17/01/2003	0.27684	0.10514
06/06/2003	0.21130	0.11799

Conclusions

When studying the Mexican short-term interest rate dynamics we find misspecification in the classical single diffusion model due to the presence of jumps. A tractable way to model such jumps is to include them in the diffusion. Good results are obtained by doing this and our evidence is justified by means of Monte Carlo simulations. Applying the Gibbs sampler we find times and sizes of jumps (with their respective probabilities). Modeling diffusion with jumps provides a better representation of the data compared with the classical diffusion.

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