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Stochastic Processes. A Critical Synthesis

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Abstract

The subject of Stochastic Processes is highly specialized and here only we present an assessment of the subject. To explain uncertainty dynamics a model is required which consists of a system et stochastic differential equation. We provide a critical synthesis of the literature and analyze the geometric behavior of a basket of useful models, stopping at computer simulation and borrowing ideas taken from the Monte Carlo method.

Key words: Wiener processes, diffusion processes, Ito formule, Ornstein-Uhlenbeck, Merton, Vasicek, Cox Ingersoll and Ross, Ho-Lee, Longstaff, Hull-White.
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Introduction

The subject of Stochastic Processes is complex, highly specialized and inmense, but nowadays is almost impossible to asses a topic without requiring some results from the theory, therefore we present a critical assessment of this vast subject and show why it is so useful.[1] A model is required when the aim is to explain uncertainty dynamics which consists of a system of stochastic differential equations, an indispensable ingredient for making apparent the reasons why the uncertain behavior appears; a good model must be able to explain this uncertainty.

Most of these works use advanced mathematics and do not provide any geometric intuition, therefore the interested reader must posses a great ability to learn by only looking at formulas. Our contribution consists in providing a critical synthesis of the literature and analyzing the geometric behavior of a basket of useful models, stopping at computer simulation and borrowing ideas taken from the Monte Carlo method. This will provide us with a very useful geometric analysis to grasp the dynamics of the stochastic processes.

More specifically, using simulations to obtain different realizations of the process which we will show in one graph, Glasserman, Paul (2004), recommends realizations and its average is taken as E[Xt]. Nevertheless, there is an inconvenience, as we have had to use many graphs, as it becomes incomprehensible to distinguish among the different paths represented by one graph. However the article has enough details so that the interested reader should be able to perform his own simulations and arrive at his own.

We first present the ideas and mathematical principles upon which the theory is built. Secondly, we talk about Wiener processes, and the very important subject of diffusion process comes with an exposition of the well known Ito Formula. To close the section there is an application to the Forward Contracts and the Black-Scholes formula to price the option contracts. Thirdly, several outstanding models are simulated: Ornstein-Uhlenbeck(1930) , Merton (1970),Vasicek (1977), CIR (1985), Ho and Lee (1986), Longstaff (1979), Hull and White (1990). Finally: Simulating Stochastic Processes in Multiple dimensions is included.

[1] One reference in Spanish is the book of Francisco Venegas, “Riesgos Financieros y Económicos”, In english a very good reference is Karatzas and Shreve.

1. The Basic Ideas

The idea of probability is rooted in Measure Theory, whose origins go back to Emile Borel and Henry Lebesgue among others. Later, in 1933, Andrey Kolmogorov would present the probability axioms.

Here we must think in terms of events and ask questions such as: has event A occurred? Or, what are our chances that the event A does not occur? In both questions we are using sets and measuring the possibility of A to occur, or to not occur.

To make the idea tractable, all the possible outcomes from the states of nature are collected in a set Ω. This set is called the sample space, and its subsets are called events. It is now of the utmost importance to see quite clearly that a set function $A \rightarrow P(A)$ is required, so that for every event A, a measure is associated that means the probability of A to occur. *Probability is a measure of events occurring in a sample space* Ω. But this set Ω must posses some properties in order to work with it, and to build a theory, we need to recall the idea of a σ-*algebra* $\Im$, which consists of a family formed by subsets of Ω that fulfill the following conditions:

1. Ω is an element of the family $\Im$.
2. If A is an element of $\Im$ the complement of A^c is again an element of $\Im$.
3. If (An) n = 1,2,… is a sequence of elements in $\Im$ then the union of all of them is again an element in $\Im$. $\bigcup_{n=1}^{\infty} A_n \in \Im$.

The idea of σ-*algebra* is critical to build a proper concept of *probability as a set function* defined on $\geq$ such that: $P:\Im \rightarrow [0,1]$; is a set map that for every event A associates its probability to occur, denoted by P(A), $0 \leq P(A) \leq 1$.

1. $P(A) \geq 0$ for every event A in $\Im$
2. $P(\Omega)=1$
3. P is σ-*additive*: if $(A_n)_{n=1}^{\infty}$ is a disjoint sequence of sets in $\Im$, fulfilling that; if $A_n \cap A_m = \sigma$ for $n \neq m$ then $P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$

The triple (Ω, $\Im$, P) is called a 'probability space'.

By a 'random variable' is meant a measurable function $X:\Omega\rightarrow\Re$ (for every open interval $(a, b)\subset\Re$, $X^{-1}(a, b)\subset\Omega$ belongs to the σ-algebra $\Im$), a technical issue is that we are going to include in the σ-algebra $\Im$ all the subsets of a zero set, that is if $A\subset B$ and $P(B)=0$ then A belongs to the family σ-algebra $\Im$ and $P(A)=0$. $P:\Im\rightarrow[0,1]$ is the probability measure defined on Ω, and the probability of an event B is defined as:

$$P_X(B) = P[X^{-1}(B)] = \int_B f_x dP$$

The function f_x is called the 'density function' of the random variable X and is the main ingredient to measure the probability of the event B to occur. The well known 'distribution function' F_X of X is the accumulated probability.

$$F_X(t) = \int_B f_x dP \quad \text{B is the set } B = (-\infty, t)$$

1.1 Conditional expectation

When two events A, B ε $\Im$ are such that $P(B)>0$ one can take the conditional probability defined as: $P(A \mid B)= P(A\cap B) / P(B)$ therefore we have the map $A\rightarrow P(A \mid B)$ and this defines a new probability measure for the family:

$B\cap\Im = \{ B\cap A \mid$ for every A $\varepsilon\Im \}$. Now to build an expected value of the random variable X over the set B, defined as: $E[X \mid B]= E[X\bullet 1_B] / P(B)$, more explicitly:

$$E[X \mid B] = [\int_B X f_X dP] / P(B)$$

Which is the expectation of X given that we lay on the event B, 1_B is the indicator function $1_B(x)=1$ when $x \varepsilon B$ and $1_B(x)=0$ when $x \varepsilon B^c$.

1.2 Random convergence

In this context, what is the meaning of a notation such as $Xn\rightarrow X$? Well there are different answers and all of them hinge on the kind of analysis one is pursuing.

The weakest concept is 'convergence in distribution', which means that one has pointwise convergence on the points of continuity of the limit function. That is: $Fn(z) \rightarrow F(z)$ for every point z such that F(z) is continuous at z, where Fn(z) is the distribution function of the random variable X_n and F(z) is the distribution function of the random variable X.

The next level is 'convergence in probability', which means that for large samples we rarely see X_n far away from X more than $\varepsilon > 0$. That is for every $\varepsilon > 0$:

$$P[\ |X_n - X| > \varepsilon\ ] \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The strongest concept is L^2 convergence, usually called 'convergence in the mean'. Here we are accepting that $E[\ |Xn|2\ ] < \infty$ and convergence means:

$$E[\ |Xn - X|^2\ ] \rightarrow 0 \text{ as } n \rightarrow \infty.$$

A 'Stochastic Processes' is a family of random variables $\{ X_t \}\ t \varepsilon T$ where each random variable is defined on the measure space $(\Omega, \Im)$. Thus we have a map:

$$X_t: W \rightarrow \Re \qquad X_t: w \rightarrow X(w,t)$$

Such that $X_t^{-1}(-\infty,z)\ \varepsilon\ \Im$ for all $z\ \varepsilon\ \Re$, for every $t \varepsilon T$ X_t is a measurable function.

The map $t \rightarrow X(w,t) : T \rightarrow \Re$ is a path. In the work every path is continuous, although it is differentiable nowhere. In other words the path of a realization is a graph that is continuous but is an infinitesimally broken line. An example of a stochastic process that has proved to be a workhorse is the Wiener process.

1.3 Wiener processes

In discrete time, we talk about white noise as a family of uncorrelated random variables, with zero mean and constant variance. In continuous time its counterpart is a Wiener process, which consists of a time path of random variables W(t) evolving as time goes by, and its motion obeys the idea of stationary and independent increments.

The most interesting analysis is for very short time intervals. We denote infinitesimal time intervals as Δt and look at the increments $\Delta W(t)$: $\Delta W(t) = W(t + \Delta t) - W(t)$ where the time interval Δt is small. The Wiener process is the essential ingredient for studying stochastic processes; concepts are generated from this notion, therefore we must formalize the following idea: a Wiener process $\{ W(t) \}_{t \geq 0}$ defined

in a probability space (Ω,F, P) is a continuous curve, that begins at the origin in which for each time t?0 it presents independent and stationary increments.

A Wiener process (called 'Standard Brownian Process' as well) in the interval [0,T] is a stochastic process $\{W(t), 0 \leq t \leq T\}$ with the following properties:

1. $W(0)= 0$.
2. Correspondence t—>W(t) is with probability one, a continuous curve in [0,T].
3. For any k and for any finite collection $0 \leq t_0 \leq t_1 \leq t_2 \leq t_3 \leq \ldots \leq t_k \leq T$, the random variables called the process increments are independent, which means: $\{W(t_1)-W(t_0), W(t_2)- W(t_1), W(t_3)-W(t_2), \ldots., W(t_k)- W(t_{k-1})\}$ are independent.
4. $W(t)-W(s)$ is distributed under the normal $N(0,t-s)$ $0 \leq s < t \leq T$.

Some consequences of the definition are that:

1. $W(t) \sim N(0,t)$ for $0 \leq t \leq T$.
2. $\Delta W(t) = \sqrt{\Delta t}\, Z$ where $Z \sim NID(0,1)$.
3. $\Delta W(t)$ is independent from $\Delta W(t+\Delta t)$.
4. $\Delta W(t)$ is independent from $W(s)=W(s)-W(0)$ for $s < t$.

See Glasserman Paul.(2004) and Karatzas and Shreve (1991). An important fact is that the continuous trajectory of a Wiener process is not differentiable, except in a set of probability zero, which means that they are not smooth curves, but infinitesimally broken.

Because $W(t) \sim N(0,t)$, we take $W(t)= (X(t) - \mu t) / \sigma$ and get $X(t) = \mu t + \sigma W(t)$.

Therefore $X(t) \sim N(\mu t, \sigma^2 t)$. Note that X(t) is the solution of the stochastic differential equation: $dX(t) = \mu dt + \sigma\, dW(t)$. For short time intervals, the Wiener process is such that, if $s< t$ then $W(t) - W(s)$ is distributed under the normal $N(0,t-s)$, in addition to the fact that these increments are independent thus we should not be concerned for a correlation structure among the increments.

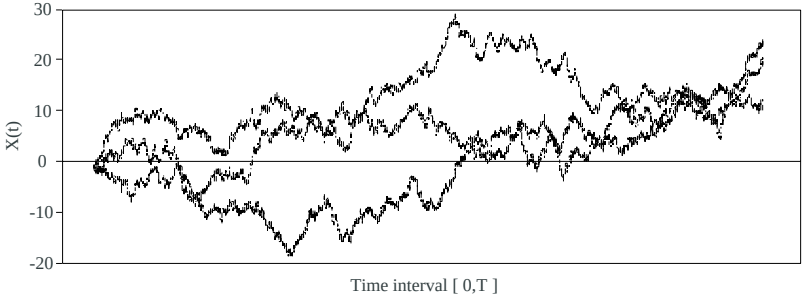
To generate a simulation of a simple Wiener processes W(t) this may be done as suggested by Glasserman (2004: 81). Take first independent standard normal variables $Z_1, Z_2, \ldots, Z_n$ and set $t_0=0$, $W(0)=0$, secondly subsequent values are taken from the recursions:

$$W(t_{i+1}) = W(t_i) + \sqrt{(t_{i+1} - t_i)} \bullet Z_{i+1} \qquad i=0,1, 2,\ldots,n-1$$

Now to generate not only W(t) but $dX(t)= \sigma\, dW(t)$ we do a small change including a given standard deviation different from unity, taking $\sigma = 13.96$ we get:

$$dX(t) = \sigma\, dW(t)$$
$$X(0) = 0$$
$$X(t_{i+1}) = X(t_i) + \sigma\sqrt{t_{i+1} - t_i}\, Z_{i+1}$$

Graph 1
Wiener processes



The Wiener process is usually taken with a drift parameter denoted by μ and its dispersion changes by a ó factor, thus its equation of motion is:

$$dX(t) = \mu dt + \sigma dW(t)$$

Where:
W(t) = is a simple Wiener process.

Values μ =0.15 y σ = 0.96 are taken. The following recursions are used:

$$X(t_{i+1}) = X(t_i) + \mu(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}\, Z_{i+1}$$

1.4 Wiener geometric process

This is a very important process because the asset yield is critical in every investment decision. The movement equation is given by:

$$dX(t) = \mu\, X(t)\, dt + \sigma\, X(t)\, dW(t)$$

Which is equivalent to:

$$dX(t)/\, X(t) = \mu\, dt + \sigma\, dW(t)$$

The component dX(t)/ X(t) is the yield. The models says that the yield moves around a mean level μ with a standard deviation of σ. The characteristics of the simple Wiener process are inherited by the yield.

To perform simulations we start at X(0) = 0.01 with the values μ =0.15 and σ = 1.96, and we use the following recursion:

$$X(t_{i+1}) = X(t_i) + \mu X(t_i)(t_{i+1} - t_i) + \sigma X(t_i)\sqrt{t_{i+1} - t_i}\, Z_{i+1}$$

Graph 2
Geometric Wiener Processes

We have to move from the Wiener process dX(t) = μdt + σ dW(t) to encompass larger class diffusion processes; the aim is to be able to manage processes by selecting a formula for: $\mu(X_t,t)$ and $\sigma(X_t,t)$ at the diffusion equation:

$$dX(t) = \mu(X_t,t)dt + \sigma(X_t,t)dW(t)$$

The Ito lemma is so important because it gives the answer to a very general class of diffusion processes by invoking a function G(x,t) with first and second continuous derivatives.

2. Diffusion Processes, the Ito equation

To exploit all the possibilities that we have at hand we need to lean on the general case which should now be considered.

Instead of thinking the relation dX(t) = μdt + σ dW(t), we want to analyze the processes:

$$dX(t) = \mu(X_t,t)dt + \sigma(X_t,t)dW(t)$$

These are called 'diffusion processes'. In order to build the simulation, the discretized version in the interval [0,T] is required. To do it, we take m length subintervals $\Delta t = T/m$ and at each subinterval $[t, t+\Delta t]$ we take an evaluation from:

$$dX(t) = \mu(X_t,t)dt+\sigma(X_t,t)dW(t)$$

After some calculations one arrives at the relation:

$$X(t+\Delta t) = X(t) + \mu(X_t,t)\,\Delta t+\sigma(X_t,t)\,\sqrt{\Delta t}\,Z, \text{ where } Z\sim NID(0,1)$$

This procedure is known in the literature as the Euler Method.

A point $X(0)=X_0$ is taken, for the moment t=0 , to be able to start the iterations and moving through the interval [0,T], use the recursions:

$$X(t+\Delta t) = X(t) + \mu(X_t,t)\,\Delta t+\sigma(X_t,t)\,\sqrt{\Delta t}\,Z, \text{ where } Z\sim NID(0,1)$$

An outstanding result in this subject is the extremely fundamental research of Kiyoshi Ito (Oksendal, 1995).

2.1 Ito lemma

Let the process: $dX_t = \mu(X_t, t)dt+\sigma(X_t,t)dW_t$ and take any function G(x,t) with the first two continuous derivatives, now define the new process $Y_t= G(X_t,t)$ then $\{Y_t\}$ is a Ito process and is given by:

$$dY_t = (\frac{\partial G}{\partial t} + \frac{\partial G}{\partial x}\mu + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}\sigma^2)dt + \frac{\partial G}{\partial x}\sigma dW_t$$

To make the calculations the following Ito rules are used:

$$dt*dt = 0$$
$$dt*dW_t = 0$$
$$dW_t*dt = 0$$
$$dW_t*dW_t = dt$$

The Ito lemma does almost all the work , the researcher only has to propose the function G(x,t) and immediately knows that:

1. The mean component that explains the displacement in the new process is:

$$\frac{\partial G}{\partial t}+\frac{\partial G}{\partial x}\mu+\frac{1}{2}\frac{\partial^2 G}{\partial X^2}\sigma^2$$

2. The variance is now:

$$\left(\frac{\partial G}{\partial x}\right)^2\sigma^2$$

To simplify matters, the discrete version of the lemma for the process X_t:

$$dX_t = \mu(X_t,t)dt+\sigma(X_t,t)dW_t$$

is given by:

$$\Delta X_t = \mu(X_t,t)\ \Delta t+\sigma(X_t,t)\ \Delta W_t$$

And for the process Y_t:

$$dY_t=(\frac{\partial G}{\partial t}+\frac{\partial G}{\partial x}\mu+\frac{1}{2}\frac{\partial^2 G}{\partial X^2}\sigma^2)dt+\frac{\partial G}{\partial x}\sigma dW_t$$

$$\Delta Y_t=(\frac{\partial G}{\partial t}+\frac{\partial G}{\partial x}\mu+\frac{1}{2}\frac{\partial^2 G}{\partial X^2}\sigma^2)\Delta t+\frac{\partial G}{\partial x}\sigma\Delta W_t$$

3. Forward Contracts and Black-Scholes

To make apparent the central place the lemma entails, we do a couple of exercises to show its practical use: with a forward contract and the well known result of Black-Scholes on options.

'Forward Contract' is given by the rule: $F = Se^{r(T-t)}$, one gets the partial derivatives very easily:

$$\frac{\partial F}{\partial S}=e^{r(T-t)} \qquad \frac{\partial^2 F}{\partial S^2}=0 \qquad \frac{\partial F}{\partial t}=-re^{(T-t)}$$

We start with a Geometric Brown model, because we have our eyes on the yield, with a expected yield of μ and a volatility of σ to specify the evolution of the spot price S_t: $dS_t = \mu S_t dt + \sigma S_t dW_t$.

Apply Ito lemma and conclude that the price of the forward contract:

$$dF = \left[e^{r(T-t)}\mu S - rSe^{r(T-t)}\right]dt + e^{r(T-t)}\sigma S dW_t$$

Simplifying we obtain:

$$dF = \left[\mu - r\right]Fdt + \sigma F dW_t$$

Easy work! It is now proved that with both the spot St and also F_t the forward price are geometric Brownian motions. S_t has an expected yield of μ, and F_t of μ-r.

3.1 Black-Scholes

A non trivial application of the Ito lemma is found in the work of Black-Scholes, we use the argument in Rue Tsay (2002, p. 232), as also in Hull (2002, p. 219, 220,221). The importance to include this material is because it exemplifies clearly a main procedure in the field: start with a known diffusion process, say a Wiener process, apply Ito´s formula and from the newly got process get a partial differential equation that its solution is the answer been searched. As the reader may see the lemma does a very important job.

Supposing that there is no arbitrage, the asset price follows a Brownian motion and the lemma is used to get the price of an option call denoted by f_t, this price depends on the spots price and is c time varying thus $f_t=f(P_t, t)$, We use the discretized version to show as well how to deal with this aspect, to be able to apply to the Ito lemma the functions: $\partial f_t/\partial t$, $\partial f_t/\partial P_t$, $\partial^2 f_t/\partial^2 t$, $\partial^2 f_t/\partial^2 P_t$, $\partial^2 f_t/\partial P_t \partial t$. All exist and are continuous functions.

The price of the asset is given by:

$$\text{A) } \Delta P_t = \mu(P_t,t)P_t\,\Delta t + \sigma(P_t,t)P_t\,\Delta W_t$$

Apply Ito lemma:

$$\text{B) } \Delta f_t = \left(\frac{\partial f_t}{\partial t} + \frac{\partial f_t}{\partial P_t}\mu P_t + \frac{1}{2}\frac{\partial^2 f_t}{\partial P_t^2}\sigma^2 P_t^2\right)\Delta t + \frac{\partial f_t}{\partial P_t}\sigma P_t \Delta W_t$$

Needless to say that the time intervals are very small (m is large).

Noting that the term ΔW_t is the same in both equations, we can look for a portfolio that does not contain the Wiener process, thus the portfolio is selected under two conditions:

It has the value of -1 e in the derivative short asset, and has the value of $\partial f_t/\partial P_t$ in the stock of the subjacent stock. «The holder of this portfolio is short one derivative security and long an amount $\partial f/\partial S$ of shares» Hull (2002, p. 220).

Denoting according to the value of the portfolio we have:

$$V_t = -f_t + (\partial f_t/\partial P_t)\, P_t$$

The change in its value ΔV_t along the moment Δt is given by:

$$\Delta V_t = -\Delta f_t + (\partial f_t/\partial P_t)\, \Delta P_t$$

Substituting from the above we have:

$$\Delta V_t = (- \partial f_t/\partial t - (1/2)(\partial^2 f_t/\partial^2 P_t)(\sigma P_t)^2\, \Delta t$$

By the way, the portfolio that has been selected in this last relation does not contain the term ΔW_t, under the hypothesis of no arbitrage, thus the portfolio is risk free along the interval Δt.

By now we have just to collect the results and we are finished,

Therefore, along the short instant of time Ät, the portfolio yields the same rate of return as any other risk free assets available in the market, so in order to fulfill the condition: $\Delta V_t = r\, V_t\, \Delta t$, *r* is the risk free rate.

Again substitute to get:

$$\Delta V_t = (-\partial f_t/\partial t - (1/2)(\partial^2 f_t/\partial^2 P_t)(\sigma P_t)^2\, \Delta t = r\, (-f_t + (\partial f_t/\partial P_t)P_t)\, \Delta t =$$

$$= r\, V_t\, \Delta t$$

$$= (\partial f_t/\partial t + (1/2)(\partial^2 f_t/\partial^2 P_t)(\sigma P_t)^2 = r\, (ft - (\partial f_t/\partial P_t)P_t)$$

We have obtained a basic partial differential equation in Black Scholes theory.

$$\frac{\partial f_t}{\partial t} + rP_t \frac{\partial f_t}{\partial P_t} + \frac{1}{2}\sigma^2 P_t^2 \frac{\partial^2 f_t}{\partial P_t^2} = rf_t$$

To solve this partial differential equation we require a boundary condition. After known procedures the solution for an European option which can be adapted for an American option also is:

$$f = \max(P_t - K,0) \quad \text{when } t=T$$

The price for the call option at the moment t is:

$$f_t = \exp[-r(T-t)\,]\, E[\max(P_T - K,0)\,]$$

It can be shown that in a neutral risk world we have:

$$f_t = exp[\,-r(T-t)]\int_K^{\infty}(P_T - K)g(P_T)dP_T$$

Here $g(P_T)$ is the density function of the random variable P_T:

$$f_t = P_t\Phi(h+) - K\,exp[T-t)]\Phi(H-)$$

$\Phi(x)$ is the normal distribution function evaluated at the point x.

$$h+ = \frac{Ln(P_t / K) + (r + \sigma^2 / 2)(T-t)}{\sigma\sqrt{T-t}}$$

$$h- = \frac{Ln(P_t / K) + (r - \sigma^2 / 2)(T-t)}{\sigma\sqrt{T-t}}$$

3.2 Simulating Stochastic Processes

Now we turn towards generating realizations of a process by computer simulations. We know that a diffusion process has two main components, the mean and the variance, which usually are time dependent.

$$dX_t = \mu(X_t, t)dt + \sigma(X_t,t)dW_t$$

Selecting formulations for the mean $\mu(X_t, t)$ and the variance $\sigma(X_t, t)$ one gets a specific diffusion process. Everyday huge amounts of new processes appear in the literature on the subject, capable of explaining any aspect of finance dynamics.

3.3 The Ornstein-Uhlenbeck Process

This process is very important in financial theory because it has an outstanding property: *mean reversion*, that is X(t), tends to oscillate around E[X(t)].

Defined as the process X(t) whose trajectory is guided by:

$$dX(t) = -\lambda X(t)dt+\sigma dW(t) \text{ where } \lambda > 0$$

[it is also may defined as $d X(t) = (m-\lambda X_t)dt+\sigma dW(t)$].

This model is used to represent assets that fluctuate around zero, because if X(t)<0 assumes negative values, the factor -? intervenes making dX(t)> 0 thus X(t) begins to grow.

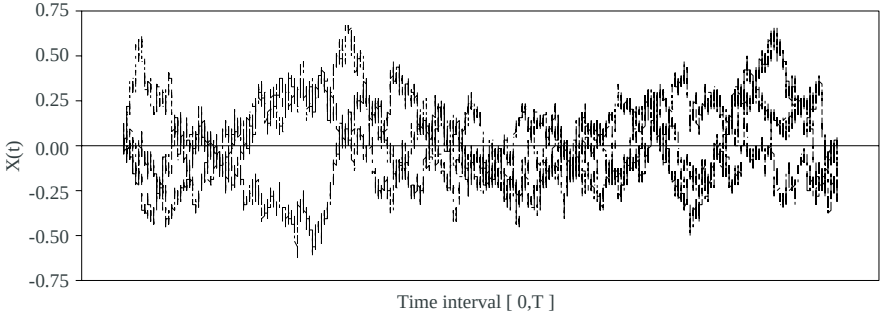
Similarly if X(t)>0 assumes positive values, the factor -λ intervenes making d X(t)< 0 thus X(t) begins to decrease; this is the central idea of the mean reversion. Consult Neftci (2000, p.271) and Gourieroux and Jasiak (2001, p. 249, 289).

The process Ornstein-Uhlenbeck has the discrete version:

$$X(t_{i+1}) =X(t_i) -\lambda X(t_i)*(t_{i+1}-t_i) +\sigma*v(t_{i+1}-t_i) *Z_i$$

With: $\lambda = 10.84$ and $\sigma = 0.96$ are taken.

Graph 3
Ornstein Ulhenbeck Processes



Now, if we take:

$$\mu(X_t,t)= a(b - X^{\alpha}_t)$$

$$\sigma(X_t,t)= \sigma X^{\beta}_t$$

We get a remarkable family of stochastic differential equations see Venegas(2006, p. 571).

$$dX(t) = a(b - X^{\alpha}_t)dt + \sigma X^{\beta}_t dW(t)$$

A group of different processes according to the alfa and beta values, the research has shown these various models which are remarkable because of their properties.

Table 1
Kit of useful Diffusion processes

Model	*Parameters*	*b*	a_t	*Process*
Merton (1970)	$\alpha = 0$ $\beta = 0$	$\mu + 1$	1	$dX_t = mdt + sdW_t$ m, s are constant
Vasicek (1977)	$\alpha = 1$ $\beta = 0$	b	a	$dX_t = a(b - r_t)dt + sdW_t$ a, b, s are constant
CIR (1985)	$\alpha = 1$ $\beta = ½$	b	a	$dX_t = a(b - X_t)dt + svX_t dW_t$ a, b, s are constant
Ho y Lee (1986)	$\alpha = 0$ $\beta = 0$	2	h_t	$dX_t = h_t dt + sdW_t$ s is constant
Longstaff (1979)	$\alpha = ½$ $\beta = ½$	b	a	$dX_t = a(b - vX_t)dt + svX_t dW_t$ a, b, s are constant
Hull and White (1990)	$\alpha = 1$ $\beta = 0$	b_t	a	$dX_t = a(b_t - X_t)dt + sdW_t$ b_t, s are a time function

Source: Venegas (2006, chap. 53: 572).

Note that Hull (1993, p.404) refers to the Hull and White model as:

$$dX(t) = (b(t)– a X(t))dt + \sigma dW(t)$$

Merton (1970)

Merton used the Wiener process denoted by ? and its dispersion changes by a ó factor, thus the equation of motion is:

$$dX(t) = \mu dt+\sigma dW(t)$$

Where:

W(t) = is a simple Wiener process; and

Values μ =0.15 y σ = 0.96 are taken.

The following recursions are used:

Graph 4
Merton Processes

Vasicek (1977)

This model has the following formulation:

$$dX(t) = a(b - X_t)dt + \sigma\, dW(t)$$

Where:
a> 0, b> 0, σ> 0 are positive constants.

This process has mean reversion, because it belongs to the Ornstein-Uhlenbeck family, specifically:

dX(t) is positive if b > X(t) thus dX(t) = a(b – X(t))dt is positive, therefore X(t) increases.
dX(t) is negative if b < X(t) thus dX(t) = a(b – X(t))dt is negative, so X(t)decreases.

The speed of convergence hinges on the parameter a, while the level where equilibrium is given by the parameter b.

In this model the long-term level b is where the process is approaching and the force with which the process X_t is being lead is the parameter a. In this model it is possible that X(t) takes negative values.

The discretized version is:

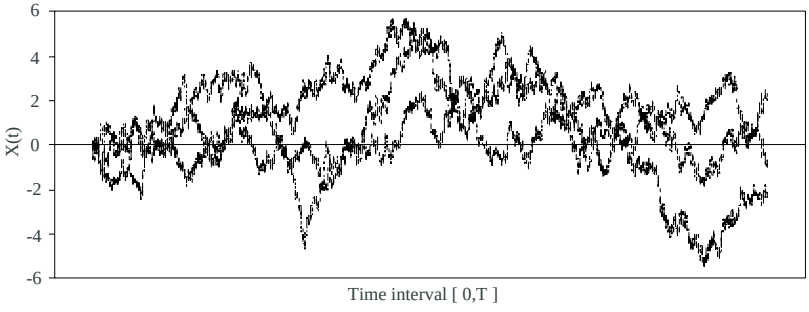
$$X(t_{i+1})=X(t_i)+ a(b - X(t_i))*(t_{i+1}-t_i) +\sigma*\sqrt{(t_{i+1}-t_i)} *Z_{i+1}$$

Where:

a, b, σ are positive constants; and

Values are taken for the simulations, a = 3.0, b = 0.5, σ = 5.4.

Graph 5
Vasicek Processes



CIR (1985)

CIR is an acronym of the Cox Ingersoll and Ross process, which is very popular in the literature.

The CIR process which is generated from the chart taking:

$$\alpha = 1\ \beta = ½\ \mu(X(t),t) = a(b - X(t)) \quad y\ \sigma(X(t),t)= \sigma\ \sqrt{X(t)}$$

$$dX_t = a(b - X_t)dt +\sigma\sqrt{X_t}\ dW_t$$

This is one of the first interest rate equilibrium structure models in continuous time of a factor that describes the rates of temporary structures. Assuming that these follow a stochastic process where their parameters are a function of themselves but they are independent in time.

This model captures the short-term interest rate dynamics with mean reversion, and it is based upon the following diffusion equation:

$$dX(t)= a(b- X(t))dt+\sigma\sqrt{X(t)}dW(t)$$

Where:
a, b, σ = are constant parameters;
a = is the force with which r_t trajectory is led towards the equilibrium level *b*.

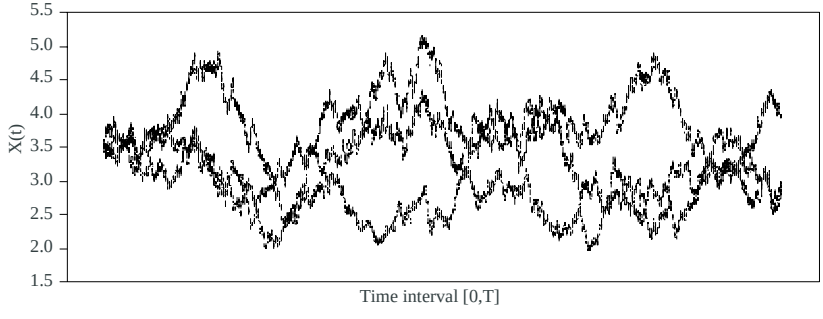
The CIR process has the property that if r(0) > 0 therefore r(t) ≥ 0 all t and also $2ab \geq \sigma^2$ therefore r(t) > 0 all t with a probability one consult Glasserman(2004, p. 120).

The most important thing provided by this model is that the term structure always generates positive interest rates, differently from the model proposed by Vasicek (1977) that can generate negative interest rates with a positive probability for some parameter values;

We consider a = 2.5, b =3.5, σ = 0.96. By using the discretization, we get:

$$X(t_{i+1}) = X(t_i) + a(b - X(t_i))(t_{i+1} - t_i)d + \sigma\sqrt{X(t_i)}\sqrt{t_{i+1} - t_i}Z_{i+1}$$

Graph 6
Cox Ingersoll Ross Processes

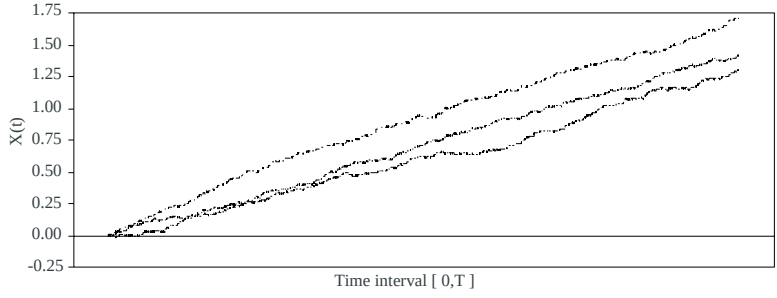


Ho y Lee (1986)

The Ho-Lee model is dX(t)=g(t)dt+σdW(t) where g(t) is a deterministic function of time. By using the discretization, we get:

$$X(t_{i+1}) = X(t_i) + G(t_i)(t_{i+1} - t_i)d + \sigma\sqrt{t_{i+1} - t_i}\,Z_i$$

Graph 7
Ho-Lee Processes



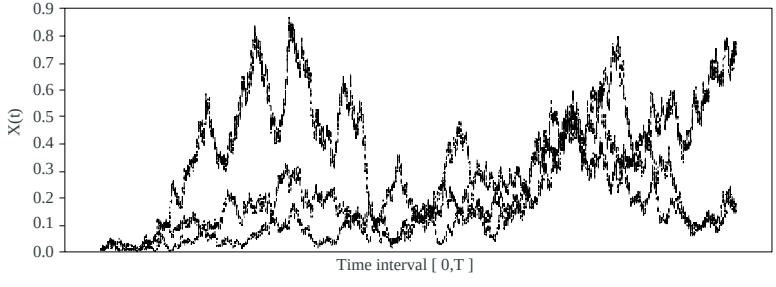
Longstaff (1979)

The Longstaff model is: dXt = a(b –vXt)dt +?vXtdWt

By using the discretization, we get:

$$X(t_{i+1}) = X(t_i) + a(b - \sqrt{X(t_i)})(t_{i+1} - t_i) + \sigma\sqrt{X(t_i)}\sqrt{t_{i+1} - t_i}\,Z_i$$

Graph 8
Longstaff Processes



Hull and White (1990)

The Hull-White process we will use, is taken from Hull (1993, 2nd edition p.404).
$dX_t = a(b_t - X_t)\,dt + \sigma dW_t$

It's known that this model can be interpreted as the Vasicek model with a time mean reversion dependent on the rate a.

$$dX(t) = (Q(t)-aX(t))dt + \sigma dW(t)$$

Where:
a , σ = are constant.

One might take Q(t) as a polynomial in t of a q grade so that:

$$Q(t)= b_0+b_1t+b_2t^2+b_3t^3+b_4t^4+\ldots+b_qt^q \qquad b_q\neq 0$$

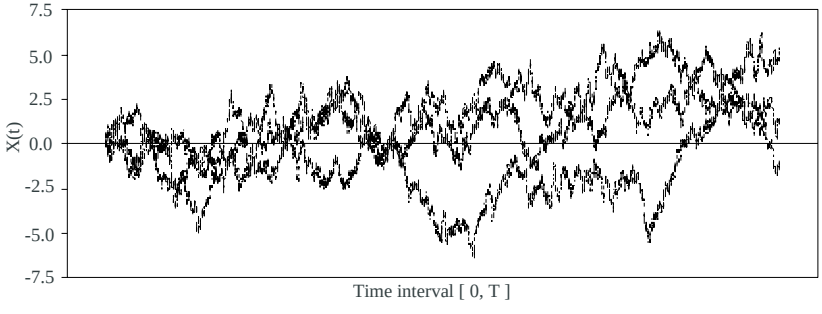
The most frequent case in applications is when q= 1, therefore we are interested in the process: $dX(t) = (b0+b1t - aX(t))\,dt + \sigma\, dW(t)$.

The discrete version for simulations is:

$$X(t_{i+1})= X(t_i)+ (b_0+b_1t_i - aX(t_i))(t_{i+1}-t_i) + \sigma*\sqrt{(t_{i+1}-t_i)}\ *Z_{i+1}$$

$a = 1.0$, $\sigma= 2.06$, $b_0= 0.1026$, $b_1 = 0.0003078$

Graph 9
Hull White Processes



4. Multiple dimensions

It is interesting to analyze a couple of examples of processes that are jointly generated under the same dynamics; they are called multivariable processes.

The most natural one is the multivariate standard Brownian motion and a second one is a multivariate Ornstein-Uhlenbeck process.

We call a vector process $W(t)= (W_1(t), W_2(t), \ldots, W_d(t))'$ a 'multivariate standard Brownian motion in $\Re^d$, if $W(0)=0$, $W(t)$ has continuous sample paths with independent stationary increments; $W(t) - W(s) \sim N(0, (t-s)\bullet I)$ $0 = s < t = T$

I is the identity dxd matrix. Note that this definition implies that each component $W_i(t)$ is a standard one dimensional Brownian motion and that $W_i(t)$ and $W_j(t)$ are independent for different values of i and j.

Now take any vector μ in $\Re^d$ and a dxd a symmetric positive definite matrix Σ. A multivariate Brownian motion in the space $\Re^d$ with drift μ and covariance matrix Σ, is given by continuous sample paths such that $X(t) - X(s) \sim N((t-s\bullet)\mu, (t-s)\bullet\Sigma)$

The seed is taken as $W(0)=0$.

Take a matrix dxd called B such that $BB' = \Sigma$

Thus the process defined as $X(t)= \mu\bullet t+B\bullet dW(t)$ is a multivariate Brownian motion, and it is possible to extend this class of processes by taking time variable parameters:

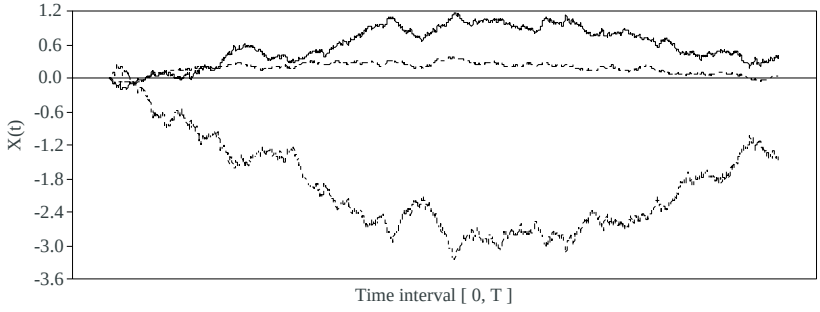
$$X(t)= \mu(t)\bullet t+B(t)\bullet dW(t) \qquad \text{note that} \qquad B(t)B(t)' = \Sigma(t)$$

To build such a process in the computer take first Z_1, Z_2, Z_3,…, Z_n independent N(0,I) in $\Re^d$, define $W(0)=0$ and take the recursions

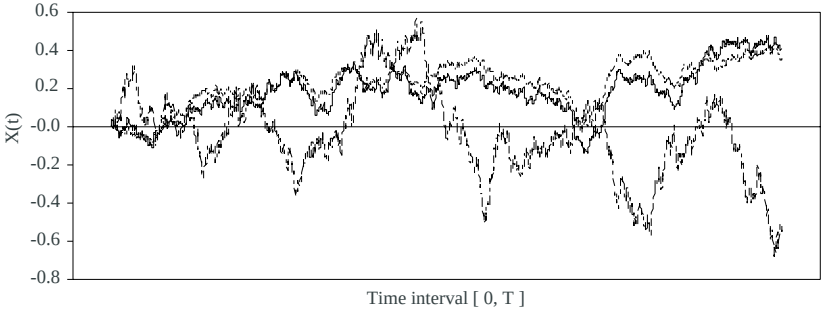
$$X(t_{i+1}) = X(t_i)+\mu\bullet(t_{i+1} - t_i)+\sqrt{(t_{i+1} - t_i)} \bullet B\bullet Z_{i+1} \text{ for } i=0,1,2,3,\ldots,n-1$$

The matrix B is such that $BB' = \Sigma$

Graph 10
Multivariate Wiener Processes



Graph 11
Multivariate Wiener Processes



The Ornstein-Uhlenbeck process is a direct extension of the univariate case.

Take a matrix dxd called B such that BB' = Σ
Define the process X(t) whose trajectory is guided by:

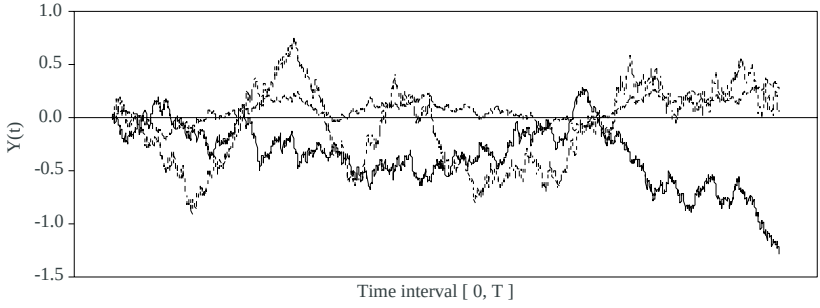
dX(t) = - λ•X(t) dt+BdW(t) where the vector ? has each component positive that is: $\lambda = (\lambda_i)$, $\lambda_i > 0$ and $\lambda \bullet X(t) = (\lambda_{i \bullet xi}(t))$ is a 'multivariate Ornstein-hlenbeck motion'.

To build such a process in the computer take first Z_1, Z_2, Z_3,…, Z_n independent N(0,I) in $\Re^d$, define W(0)=0 and take the recursions:

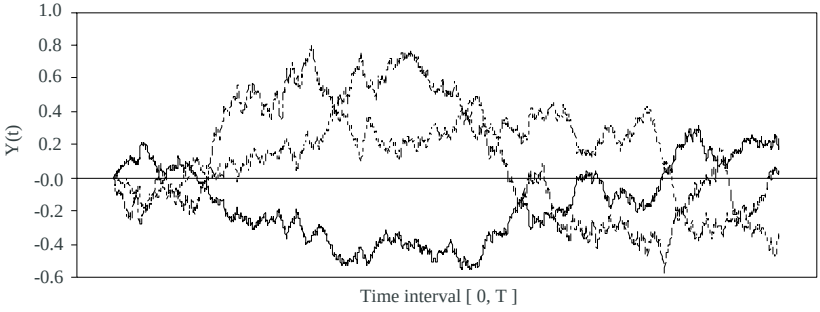
$$X(t_{i+1}) = X(t_i) - \lambda \bullet X(t_i) \bullet (t_{i+1} - t_i) + \sqrt{(t_{i+1} - t_i)} \bullet B \bullet Z_{i+1} \quad \text{for } i=0,1,2,3,\ldots,n-1$$

The matrix B is such that $BB' = \Sigma$

Graph 12
Multivariate Ornstein-Uhlenbeck Processes



Graph 13
Multivariate Ornstein-Uhlenbeck Processes



Conclusions

Stochastic processes is a rich field with ample possibilities, although there is cost to enter into the topic: high level in the quantitative skill and knowledge in the subject of modeling and simulating a model using the computer, the reward is a much more deep understanding of what is «Stochastics». Finance requires these notions in which chance and random behavior occur every day, but as well Macroeconomics nowadays is modeled by using the notion of probable outcomes and

erratic phenomena. Therefore the scholar in Finance Economics is compelled to a much more higher level in Mathematics than ever before. We have seen that the natural path to knowledge is to rely in geometric visualizations to understand probabilistic dynamics. Is as Mankiw(2006) says: an economist is a (social) scientific and a (social) engineer, neither side dominates the other, a proper balance is required.

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